Quantized Affine Algebras and Crystals with Head

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Abstract

Motivated by the work of Nakayashiki on the inhomogeneous vertex models of 6-vertex type, we introduce the notion of crystals with head. We show that the tensor product of the highest weight crystal $B(\lambda)$ of level k and the perfect crystal B_l of level l is isomorphic to the tensor product of the perfect crystal B_{l-k} of level $l-k$ and the highest weight crystal $B(\lambda')$ of level k.

1 Introduction

In [9], Nakayashiki studied the inhomogeneous vertex models of 6-vertex type, and he explained the degeneration of the ground states from the point of view of the representation theory as follows. Let $V(\Lambda_i)$ be the irreducible $U_q'(\widehat{\mathfrak{sl}}_2)$-module with the highest weight Λ_i $(i = 0, 1)$ of level 1, and let V_s be the $(s+1)$-dimensional $U_q'(\widehat{\mathfrak{sl}}_2)$-module. Then there exists an intertwiner

$$\Phi(z) : (V_{s-1})_z \otimes V(\Lambda_i) \to V(\Lambda_{i+1}) \otimes (V_s)_z.$$

He identified $(V_{s-1})_z$ with the degeneration of the ground states.

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The $q = 0$ limit can be described in terms of crystal bases. Let B_s be the crystal base of V_s, and let $B(\Lambda_i)$ be the crystal base of $V(\Lambda_i)$. Then we have an isomorphism of crystals

$$B_{s-1} \otimes B(\Lambda_i) \cong B(\Lambda_{i+1}) \otimes B_s.$$

The purpose of this paper is to generalize the above result on crystals in a more general situation, replacing $U_q'(\widehat{\mathfrak{sl}}_2)$ with quantized affine algebras $U_q'(\mathfrak{g})$, $B(\Lambda_i)$ with the crystals of the integrable highest weight representations of arbitrary positive level, and B_s with perfect crystals.

The crystal of the integrable highest weight representation has a unique highest weight vector. Namely, it contains a unique vector b such that $\tilde{e}_i b = 0$ for all i and all the other vectors can be obtained from b by applying $\tilde{f}_i$'s successively. However, neither $B_{s-1} \otimes B(\Lambda_i)$ nor $B(\Lambda_{i+1}) \otimes B_s$ has such properties. Instead, they satisfy weaker properties: the highest weight vector has to be replaced with a subset consisting of several vectors, which we call the *head*. This is a combinatorial phenomenon corresponding to the degeneration of the ground states in the exactly solvable models.

Let B be a crystal. For $b \in B$, let $E(b)$ be the smallest subset of B containing b and stable under the $\tilde{e}_i$'s. We say that B has a *head* if $E(b)$ is a finite set for any $b \in B$. For such a crystal, we define its head $H(B)$ to be $\{b \in B \mid E(b') = E(b) \text{ for every } b' \in E(b)\}$. Then the head replaces the role of highest weight vectors: all the vectors in B can be obtained from vectors in the head by applying $\tilde{f}_i$'s successively.

If D is a finite regular crystal and $B(\lambda)$ is the crystal of the integrable highest weight representation with highest weight λ of level k, then $D \otimes B(\lambda)$ has a head and its head is given by $D \otimes u_\lambda$, where u_λ is the highest weight vector of $B(\lambda)$. However, if we change the order of the tensor product, the situation is completely different. The crystal $B(\lambda) \otimes D$ has a head, but $u_\lambda \otimes D$ is not the head in general. In this paper, we prove that, for a perfect crystal B_l of level $l > k$, $B(\lambda) \otimes B_l$ is isomorphic to the crystal $B_{l-k} \otimes B(\lambda')$ for another dominant integral weight λ' of level k and the perfect crystal B_{l-k} of level $l - k$ (see Theorem 5.4 for more precise statements).

The proof is based on the theory of *coherent families of perfect crystals* developed in [5] and the characterization of crystals of the form $D \otimes B(\lambda)$. We introduce the notion of *regular head* (Definition 4.1), and we prove that any connected regular crystal with regular head is isomorphic to a crystal of the form $H(B) \otimes B(\lambda)$ for some dominant integral weight λ (see Theorem 4.7 for more precise statements). Then, we check the regularity condition for the coherent families of perfect crystals.

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2 Quantized Affine Algebras

Let I be a finite index set and $A = (a_{ij})_{i,j\in I}$ a generalized Cartan matrix of affine type. We choose a vector space $\mathfrak{t}$ of dimension $|I|+1$, and let $\Pi = \{\alpha_i | \ i \in I\}$ and $\Pi^\vee = \{h_i | \ i \in I\}$ be linearly independent subsets of $\mathfrak{t}^*$ and $\mathfrak{t}$, respectively, satisfying $\langle h_i, \alpha_j\rangle = a_{ij}$ for all $i, j \in I$. The α_i (resp. h_i) are called the *simple roots* (resp. *simple coroots*), and the free abelian group $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ (resp. $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}h_i$) is called the *root lattice* (resp. *dual root lattice*). We denote by $\delta = \sum_{i\in I} a_i\alpha_i \in Q$ the smallest positive *imaginary root* and $c = \sum_{i\in I} a_i^\vee h_i \in Q^\vee$ the *canonical central element* (cf. [2, Chapter 6]). Set $\mathfrak{t}^*_{\mathrm{cl}} = \mathfrak{t}^*/\mathbb{C}\delta$ and let $\mathrm{cl} : \mathfrak{t}^* \to \mathfrak{t}^*_{\mathrm{cl}}$ be the canonical projection. We denote by $\mathfrak{t}^{*0} = \{\lambda \in \mathfrak{t}^* | \ \langle c, \lambda\rangle = 0\}$ and $\mathfrak{t}^{*0}_{\mathrm{cl}} = \mathrm{cl}(\mathfrak{t}^{*0})$.

Let $P = \{\lambda \in \mathfrak{t}^* \mid \langle h_i, \lambda\rangle \in \mathbb{Z} \ \text{ for all } i \in I\}$ be the *weight lattice* and $P^\vee = \{h \in \mathfrak{t} \mid \langle h, \alpha_i\rangle \in \mathbb{Z} \ \text{ for all } i \in I\}$ be the *dual weight lattice*. Note that $\alpha_i, \Lambda_i \in P$ and $h_i \in P^\vee$, where $\Lambda_i \in \mathfrak{t}^*$ are linear forms satisfying $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ $(i, j \in I)$. Set $P_{\mathrm{cl}} = \mathrm{cl}(P) = \mathrm{Hom}(Q^\vee, \mathbb{Z}) \subset \mathfrak{t}^*_{\mathrm{cl}}$, $P^0 = \{\lambda \in P | \ \langle c, \lambda\rangle = 0\} \subset \mathfrak{t}^{*0}$, and $P^0_{\mathrm{cl}} = \mathrm{cl}(P^0) \subset \mathfrak{t}^{*0}_{\mathrm{cl}}$.

Since the generalized Cartan matrix A is symmetrizable, there is a non-degenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{t}^*$ satisfying

$$\langle h_i, \lambda\rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \ \text{ for all } i \in I, \ \lambda \in \mathfrak{t}^*.$$

We normalize the bilinear form so that we have

$$(\delta, \lambda) = \langle c, \lambda\rangle.$$

Note that $\mathfrak{t}^{*0}_{\mathrm{cl}}$ has a non-degenerate symmetric bilinear form induced by that on $\mathfrak{t}^*$. We take the smallest positive integer γ such that $\gamma(\alpha_i, \alpha_i)/2$ is a positive integer for all $i \in I$.

Definition 2.1 The *quantized affine algebra* $U_q(\mathfrak{g})$ is the associative algebra with 1 over $\mathbb{C}(q^{1/\gamma})$ generated by the elements e_i, f_i $(i \in I)$ and $q(h)$ $(h \in \gamma^{-1}P^\vee)$ satisfying the following defining relations:

$$\begin{aligned}
&q(0) = 1, \ q(h)q(h') = q(h + h') \quad (h, h' \in \gamma^{-1}P^\vee),\\
&q(h)e_iq(-h) = q^{\langle h,\alpha_i\rangle}e_i,\\
&q(h)f_iq(-h) = q^{-\langle h,\alpha_i\rangle}f_i \quad (h \in \gamma^{-1}P^\vee, i \in I),\\
&[e_i, f_j] = \delta_{ij}\frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \quad (i, j \in I),\\
&\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{ij}-k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)} = 0 \ \ (i \neq j),
\end{aligned} \tag{2.1}$$

where $q_i = q^{(\alpha_i,\alpha_i)/2}$, $t_i = q(\frac{(\alpha_i,\alpha_i)}{2}h_i)$, $e_i^{(k)} = e_i^k/[k]_i!$, $f_i^{(k)} = f_i^k/[k]_i!$, $[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}$, and $[k]_i! = [1]_i[2]_i \dots [k]_i$ for all $i \in I$.

The quantized affine algebra $U_q(\mathfrak{g})$ has a Hopf algebra structure with comultiplication Δ, counit ε, and antipode S defined by

$$
\begin{aligned}
&\Delta(q(h)) = q(h) \otimes q(h), \\
&\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i, \\
&\varepsilon(q(h)) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0, \\
&S(q(h)) = q(-h), \quad S(e_i) = -e_i t_i, \quad S(f_i) = -t_i^{-1} f_i
\end{aligned}
\tag{2.2}
$$

for all $h \in \gamma^{-1}P^\vee$, $i \in I$.

We denote by $U_q'(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i $(i \in I)$ and $q(h)$ $(h \in \gamma^{-1}Q^\vee)$, which will also be called the *quantized affine algebra*.

A $U_q'(\mathfrak{g})$-module M is called *integrable* if it has the *weight space decomposition* $M = \bigoplus_{\lambda \in P_{\mathrm{cl}}} M_\lambda$, where $M_\lambda = \{u \in M |\ q(h)u = q^{\langle h,\lambda\rangle}u \ \text{ for all } h \in \gamma^{-1}Q^\vee\}$, and M is $U_q'(\mathfrak{g})_i$-locally finite (i.e., $\dim U_q'(\mathfrak{g})_i u < \infty$ for all $u \in M$) for all $i \in I$, where $U_q'(\mathfrak{g})_i$ denotes the subalgebra of $U_q'(\mathfrak{g})$ generated by e_i, f_i, and t_i.

3 Crystals with Head

In studying the structure of integrable representations of quantized affine algebras, the *crystal base theory* developed in [3] provides a very powerful combinatorial method. In this section, we develop the theory of *crystals with head*. We first recall the definition of *crystals* given in [4].

Definition 3.1 A *crystal* B is a set together with the maps $\mathrm{wt} : B \to P$, $\varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}$, $\varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\}$, $\tilde{e}_i : B \to B \sqcup \{0\}$, $\tilde{f}_i : B \to B \sqcup \{0\}$ $(i \in I)$ satisfying the axioms:

$$
\begin{aligned}
&\langle h_i, \mathrm{wt}(b)\rangle = \varphi_i(b) - \varepsilon_i(b) \ \text{ for all } b \in B, \\
&\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i \ \text{ for } b \in B \ \text{ with } \tilde{e}_i b \in B, \\
&\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i \ \text{ for } b \in B \ \text{ with } \tilde{f}_i b \in B, \\
&\tilde{f}_i b = b' \ \text{ if and only if } \ b = \tilde{e}_i b' \ \text{ for } b, b' \in B, \\
&\tilde{e}_i b = \tilde{f}_i b = 0 \ \text{ if } \ \varepsilon_i(b) = -\infty.
\end{aligned}
\tag{3.1}
$$

Definition 3.2 For two crystals B_1 and B_2, a *morphism* of crystals from B_1 to B_2 is a map $\psi : B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ such that

$$
\begin{aligned}
&\psi(0) = 0,\\
&\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b) \ \text{ for } b \in B_1 \ \text{ with } \ \tilde{e}_i b \in B_1,\ \psi(b) \in B_2,\ \psi(\tilde{e}_i b) \in B_2,\\
&\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \ \text{ for } b \in B_1 \ \text{ with } \ \tilde{f}_i b \in B_1,\ \psi(b) \in B_2,\ \psi(\tilde{f}_i b) \in B_2,\\
&\mathrm{wt}(\psi(b)) = \mathrm{wt}(b) \ \text{ for } b \in B_1 \ \text{ with } \ \psi(b) \in B_2,\\
&\varepsilon_i(\psi(b)) = \varepsilon_i(b),\ \ \varphi_i(\psi(b)) = \varphi_i(b) \ \text{ for } b \in B_1 \ \text{ with } \ \psi(b) \in B_2.
\end{aligned}
\tag{3.2}
$$

A morphism $\psi : B_1 \to B_2$ is called an *embedding* if the map $\psi : B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ is injective. In this case, we call B_1 a *subcrystal* of B_2.

For two crystals B_1 and B_2, we define their *tensor product* $B_1 \otimes B_2$ as follows. The underlying set is $B_1 \times B_2$. For $b_1 \in B_1$, $b_2 \in B_2$, we write $b_1 \otimes b_2$ for (b_1, b_2) and we understand $b_1 \otimes 0 = 0 \otimes b_2 = 0$. We define the maps $\mathrm{wt} : B_1 \otimes B_2 \to P$, $\varepsilon_i : B_1 \otimes B_2 \to \mathbb{Z} \sqcup \{-\infty\}$, $\varphi_i : B_1 \otimes B_2 \to \mathbb{Z} \sqcup \{-\infty\}$, $\tilde{e}_i : B_1 \otimes B_2 \to B_1 \otimes B_2 \sqcup \{0\}$, $\tilde{f}_i : B_1 \otimes B_2 \to B_1 \otimes B_2 \sqcup \{0\}$ $(i \in I)$ as follows:

$$
\begin{aligned}
&\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),\\
&\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1),\ \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle),\\
&\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2),\ \varphi_i(b_1) + \langle h_i, \mathrm{wt}(b_2)\rangle),\\
&\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2),\\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2),\end{cases}\\
&\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2),\\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2).\end{cases}
\end{aligned}
\tag{3.3}
$$

In the sequel, we will only consider the crystals over the quantized affine algebra $U_q'(\mathfrak{g})$. Hence the weights of crystals will be elements of P_{cl}. For example, for $\lambda \in P_{\mathrm{cl}}$, consider the set $T_\lambda = \{t_\lambda\}$ with one element. Define $\mathrm{wt}(t_\lambda) = \lambda$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$, and $\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0$ $(i \in I)$. Then T_λ is a crystal and we have $T_\lambda \otimes T_{\lambda'} \cong T_{\lambda+\lambda'}$.

For a dominant integral weight λ, we denote by $B(\lambda)$ the crystal associated with the integrable highest weight representation with highest weight λ, and u_λ the highest weight vector of $B(\lambda)$. The highest weight vector u_λ is the unique element of $B(\lambda)$ with weight λ satisfying $\tilde{e}_i u_\lambda = 0$ for all $i \in I$.

For a subset J of I, we denote by $U_q'(\mathfrak{g}_J)$ the subalgebra of $U_q'(\mathfrak{g})$ generated by e_i, f_i, and t_i $(i \in J)$. Note that if $J \subsetneq I$, then $\mathfrak{g}_J$ is a finite-dimensional semisimple Lie algebra. Similarly, for a subset J of I, we denote by B_J the crystal B equipped with the maps wt, ε_i, φ_i, $\tilde{e}_i$, and $\tilde{f}_i$ for $i \in J$. We say that a crystal B over $U_q'(\mathfrak{g})$ is *regular* if, for any $J \subsetneq I$, B_J is isomorphic to the crystal associated with an integrable $U_q'(\mathfrak{g}_J)$-module. This

condition is equivalent to saying that the same assertion holds for any $J \subsetneq I$ with one or two elements (see [6, Proposition 2.4.4]).

Let B be a regular crystal. For $b \in B$, let $\tilde{e}_i^{\max} b = \tilde{e}_i^k b$ such that $\tilde{e}_i^k b \neq 0$, $\tilde{e}_i^{k+1} b = 0$, and define

$$E(b) = \{\tilde{e}_{i_1} \dots \tilde{e}_{i_l} b \mid l \geq 0 \text{ and } i_1, \dots, i_l \in I\} \setminus \{0\},$$
$$E^{\max}(b) = \{\tilde{e}_{i_1}^{\max} \dots \tilde{e}_{i_l}^{\max} b \mid l \geq 0 \text{ and } i_1, \dots, i_l \in I\}. \tag{3.4}$$

It follows that

$$\begin{aligned} E^{\max}(b) &\subset E(b), \\ E(b') &\subset E(b) \quad \text{for all } b' \in E(b), \\ E^{\max}(b') &\subset E^{\max}(b) \quad \text{for all } b' \in E^{\max}(b). \end{aligned} \tag{3.5}$$

Recall that the Weyl group W acts on the regular crystals ([4]). For each $i \in I$, the simple reflection s_i acts on the regular crystal B by

$$S_i(b) = \begin{cases} \tilde{f}_i^{\langle h_i, \mathrm{wt}(b) \rangle} b & \text{if } \langle h_i, \mathrm{wt}(b) \rangle \geq 0, \\ \tilde{e}_i^{-\langle h_i, \mathrm{wt}(b) \rangle} b & \text{if } \langle h_i, \mathrm{wt}(b) \rangle \leq 0. \end{cases} \tag{3.6}$$

For $w = s_{i_r} s_{i_{r-1}} \dots s_{i_1} \in W$, its action is given by $S_w = S_{i_r} S_{i_{r-1}} \dots S_{i_1}$.

We first prove:

Lemma 3.3 *Let B be a finite regular crystal.*

(a) *We have $E(S_w(b)) = E(b)$ for all $b \in B$, $w \in W$.*

(b) *$E(b)$ is a connected component of B for any $b \in B$.*

Proof. (a) It suffices to show that $S_i(b) \in E(b)$ for all $b \in B$, $i \in I$. If $\lambda = \mathrm{wt}(b)$ satisfies $\langle h_i, \lambda \rangle \leq 0$, then by (3.6), our assertion is obvious. If $\langle h_i, \lambda \rangle > 0$, take $w = s_{i_l} \dots s_{i_1} \in W$ such that $\langle h_{i_k}, s_{i_{k-1}} \dots s_{i_1} \lambda \rangle < 0$ for $k = 1, \dots, l$ and $s_i \lambda = w\lambda$ (see [1, Lemma 1.4]). Then for each $n \geq 1$, we have $S_w (S_i S_w)^n b \in E(b)$. Since $S_i S_w$ has finite order, there exists $n > 0$ such that $(S_i S_w)^n b = b$. Hence $S_i b = S_w (S_i S_w)^{n-1} b \in E(b)$.

(b) Note that for any $b \in B$, we have $\tilde{f}_i b = \tilde{e}_i^{\varphi_i(b)-1} S_i(\tilde{e}_i^{\max} b)$. By (a), this implies $E(b)$ is stable under $\tilde{f}_i$ for all $i \in I$. Hence we have the desired result. Q.E.D.

Definition 3.4 We say that a regular crystal B has a *head* if $E(b)$ is a finite set for any $b \in B$. In this case, we define the *head* $H(B)$ of B to be

$$H(B) = \{b \in B \mid E(b') = E(b) \quad \text{for every } b' \in E(b)\}, \tag{3.7}$$

and B is called a *crystal with head*.

In the following, we prove some of the basic properties of the crystals with head.

Lemma 3.5 *Suppose that B has a head $H(B)$.*

(a) *The head $H(B)$ is stable under $\tilde{e}_i$'s ($i \in I$).*

(b) *$E(b) \cap H(B) \neq \emptyset$ for all $b \in B$.*

(c) *If $b \in H(B)$, then either $\tilde{e}_i b = 0$ for all $i \in I$ or there exist $i_1, \ldots, i_l \in I$ ($l \geq 1$) such that $b = \tilde{e}_{i_l} \ldots \tilde{e}_{i_1} b$.*

Proof. (a) If $b \in H(B)$, then $E(b) \subset H(B)$, since for $b' \in E(b)$ and $b'' \in E(b') \subset E(b)$, we have $E(b'') = E(b) = E(b')$.

(b) For any $b \in B$, take $b' \in E(b)$ such that $E(b')$ has the smallest cardinality. Then, since $E(b'') \subset E(b')$ for any $b'' \in E(b') \subset E(b)$, we have $E(b'') = E(b')$, which implies b' belongs to $H(B)$.

(c) If $b \in H(B)$ and $\tilde{e}_{i_1} b \neq 0$ for some $i_1 \in I$, then by definition we have $E(b) = E(\tilde{e}_{i_1} b)$. Then $b \in E(b)$ implies $b = \tilde{e}_{i_l} \ldots \tilde{e}_{i_1} b$ for some $i_2, \ldots, i_l \in I$. Q.E.D.

Lemma 3.6 *Let B be a regular crystal with head and H a subset of B.*

(a) *If H is stable under $\tilde{e}_i$'s ($i \in I$) and $E(b) \cap H \neq \emptyset$ for any $b \in B$, then $H(B)$ is contained in H.*

(b) *If, in addition, $E(b) = E(b')$ for any $b \in H$ and $b' \in E(b)$, then $H = H(B)$.*

Proof. (a) If $b \in H(B)$, take $b' \in E(b) \cap H$. Then $b \in E(b) = E(b') \subset H$.

(b) If $b \in H$ and $b' \in E(b) \cap H(B)$, then $b \in E(b) = E(b') \subset H(B)$. Q.E.D.

Corollary 3.7 *If B is a finite regular crystal, then $H(B) = B$.*

Proof. We may assume that B is connected. By Lemma 3.3, we have $E(b) = B$ for all $b \in B$. Hence $H(B) = B$. Q.E.D.

4 Structure of Crystals with Head

Let $\psi : H(B) \hookrightarrow B$ denote the inclusion map.

Definition 4.1 We say that B has a *regular head* if the head $H(B)$ of B becomes a regular crystal with the maps wt, ε_i, φ_i, $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ defined by

$$
\begin{aligned}
\tilde{e}_i b &= \psi^{-1}(\tilde{e}_i \psi(b)), \\
\tilde{f}_i b &= \begin{cases} \psi^{-1}(\tilde{f}_i \psi(b)) & \text{if } \tilde{f}_i \psi(b) \in H(B), \\ 0 & \text{otherwise,} \end{cases} \\
\varepsilon_i(b) &= \varepsilon_i(\psi(b)), \\
\varphi_i(b) &= \max\{k \geq 0 |\ \tilde{f}_i^k b \in H(B)\} \\
&= \max\{k \geq 0 |\ b \in \tilde{e}_i^k H(B)\}, \\
\mathrm{wt}(b) &= \sum_i (\varphi_i(b) - \varepsilon_i(b))\Lambda_i \in P_{\mathrm{cl}}.
\end{aligned}
\tag{4.1}
$$

Let $b \in H(B)$. Then $E(b) \subset H(B)$. If $b' \in E(b)$ satisfies $\tilde{f}_i b' \in H(B)$, then $\tilde{f}_i b' \in E(\tilde{f}_i b') = E(\tilde{e}_i \tilde{f}_i b') = E(b') = E(b)$, and hence $E(b)$ is stable under $\tilde{f}_i$'s $(i \in I)$. Therefore the connected components of $H(B)$ are of the form $E(b)$.

Let $E(b_0)$ be a connected component of $H(B)$ and set $W(b) = \mathrm{wt}(\psi(b)) - \mathrm{wt}(b)$ for $b \in E(b_0)$, where $\psi : H(B) \hookrightarrow B$ is the inclusion map. Note that, for all $i, j \in I$, we have

$$
\begin{aligned}
\langle h_i, W(\tilde{e}_j b)\rangle &= \langle h_i, \mathrm{wt}(\psi(\tilde{e}_j b))\rangle - \langle h_i, \mathrm{wt}(\tilde{e}_j b)\rangle \\
&= \langle h_i, \mathrm{wt}(\psi(b)) + \alpha_j\rangle - \langle h_i, \mathrm{wt}(b) + \alpha_j\rangle \\
&= \langle h_i, \mathrm{wt}(\psi(b)) - \mathrm{wt}(b)\rangle = \langle h_i, W(b)\rangle.
\end{aligned}
$$

Hence, $W(\tilde{e}_j b) = W(b)$ for all $j \in I$, which implies $W(b)$ is constant on $E(b_0)$.

Let $\lambda_0 = \mathrm{wt}(\psi(b_0)) - \mathrm{wt}(b_0)$. Since

$$
\begin{aligned}
\langle h_i, \lambda_0\rangle &= \langle h_i, \mathrm{wt}(\psi(b_0)) - \mathrm{wt}(b_0)\rangle \\
&= \varphi_i(\psi(b_0)) - \varphi_i(b_0) \geq 0,
\end{aligned}
$$

λ_0 is dominant integral. We will show that there exists a unique embedding of regular crystals $E(b_0) \otimes B(\lambda_0) \to B$ sending $b \otimes u_{\lambda_0}$ to $\psi(b)$ for all $b \in E(b_0)$, where u_{λ_0} is the highest weight vector of $B(\lambda_0)$.

Let D be a finite regular crystal, and let λ be a dominant integral weight. We denote by $B(\lambda)$ the crystal associated with the integrable highest weight $U'_q(\mathfrak{g})$-module $V(\lambda)$ with highest weight λ, and let u_λ be the highest weight vector of $B(\lambda)$.

Lemma 4.2 *For any $b \in D \otimes B(\lambda)$, we have*

$$E^{\max}(b) \cap (D \otimes u_\lambda) \neq \emptyset.$$

Proof. If it were not true, there would exist $b = b_1 \otimes b_2 \in D \otimes B(\lambda)$ such that $E^{\max}(b) \subset D \otimes b_2$ and $b_2 \in B(\lambda) \setminus \{u_\lambda\}$. By the tensor product rule, this implies $E^{\max}(b) = E^{\max}(b_1) \otimes b_2$. Since $b_2 \neq u_\lambda$, there exists $i \in I$ such that $\varepsilon_i(b_2) > 0$. Take $b' \in E^{\max}(b_1)$ such that $\varphi_i(b') = 0$. Such a b' exists by [1, Lemma 1.5]. Then we have $\tilde{e}_i(b' \otimes b_2) = b' \otimes (\tilde{e}_i b_2)$, which contradicts $\tilde{e}_i^{\max}(b' \otimes b_2) \in D \otimes b_2$. Q.E.D.

Lemma 4.3 *The regular crystal $D \otimes B(\lambda)$ has a regular head and $H(D \otimes B(\lambda)) = D \otimes u_\lambda$, which is isomorphic to D as a crystal.*

Proof. Since $E(b_1 \otimes b_2) \subset E(b_1) \otimes E(b_2)$, $D \otimes B(\lambda)$ has a head. The second assertion follows from Lemma 3.3, Lemma 3.6 and Lemma 4.2. Hence $D \otimes B(\lambda)$ has a regular head. Q.E.D.

Proposition 4.4 *Let D be a finite regular crystal, and let λ be a dominant integral weight. Then for every $b \in D \otimes B(\lambda)$, there exists a positive integer N such that $\tilde{e}_{i_N}^{\max} \dots \tilde{e}_{i_1}^{\max} b \in D \otimes u_\lambda$ if $\tilde{e}_{i_k}^{\max} \dots \tilde{e}_{i_1}^{\max} b \neq \tilde{e}_{i_{k-1}}^{\max} \dots \tilde{e}_{i_1}^{\max} b$ for $1 \le k \le N$.*

Proof. If the proposition were false, there would exist $b \in (D \otimes B(\lambda)) \setminus (D \otimes u_\lambda)$ and $l > 0$ such that

$$(4.2) \qquad b = \tilde{e}_{i_l}^{\max} \dots \tilde{e}_{i_1}^{\max} b \quad \text{and} \quad \tilde{e}_{i_k}^{\max} \dots \tilde{e}_{i_1}^{\max} b \neq \tilde{e}_{i_{k-1}}^{\max} \dots \tilde{e}_{i_1}^{\max} b \ \text{ for } \ k = 1, \dots, l.$$

Set $\tilde{e}_{i_k}^{\max} \dots \tilde{e}_{i_1}^{\max} b = b_k \otimes b'$ with $b_k \in D$ and $b' \in B(\lambda)$. Then b' does not depend on k and we have $b_k = \tilde{e}_{i_k}^{\max} b_{k-1}$. Since D is a finite crystal, all of its weights have level 0. Hence the square lengths of its weights are well-defined.

Since $\mathrm{wt}(b_k) = \mathrm{wt}(b_{k-1}) + \varepsilon_{i_k}(b_{k-1})\alpha_{i_k}$, we have

$$(4.3) \quad \begin{aligned} (\mathrm{wt}(b_k), \mathrm{wt}(b_k)) &= (\mathrm{wt}(b_{k-1}), \mathrm{wt}(b_{k-1})) + 2\varepsilon_{i_k}(b_{k-1})(\mathrm{wt}(b_{k-1}), \alpha_{i_k}) \\ &\quad + \varepsilon_{i_k}(b_{k-1})^2(\alpha_{i_k}, \alpha_{i_k}) \\ &= (\mathrm{wt}(b_{k-1}), \mathrm{wt}(b_{k-1})) + \varepsilon_{i_k}(b_{k-1})(\alpha_{i_k}, \alpha_{i_k})\langle h_{i_k}, \mathrm{wt}(b_{k-1})\rangle \\ &\quad + \varepsilon_{i_k}(b_{k-1})^2(\alpha_{i_k}, \alpha_{i_k}) \\ &= (\mathrm{wt}(b_{k-1}), \mathrm{wt}(b_{k-1})) + (\alpha_{i_k}, \alpha_{i_k})\varepsilon_{i_k}(b_{k-1})\varphi_{i_k}(b_{k-1}) \\ &\ge (\mathrm{wt}(b_{k-1}), \mathrm{wt}(b_{k-1})) \end{aligned}$$

for all $k \ge 1$. Hence $(\mathrm{wt}(b_k), \mathrm{wt}(b_k))$ are the same for all $k \ge 1$. Since (4.3) is the equality and $\varepsilon_{i_k}(b_{k-1}) > 0$, we have $\varphi_{i_k}(b_{k-1}) = 0$. Since $\tilde{e}_{i_k}(b_{k-1} \otimes b') = \tilde{e}_{i_k} b_{k-1} \otimes b'$, we have $\varphi_{i_k}(b_{k-1}) \ge \varepsilon_{i_k}(b')$, and hence $\varepsilon_{i_k}(b') = 0$. Write $\mathrm{wt}(\tilde{e}_{i_l}^{\max} \dots \tilde{e}_{i_1}^{\max} b) = \mathrm{cl}(t_1\alpha_{i_1} + \cdots + t_l\alpha_{i_l}) + \mathrm{wt}(b)$. Since $\mathrm{wt}(b) = \mathrm{wt}(\tilde{e}_{i_l}^{\max} \dots \tilde{e}_{i_1}^{\max} b)$, $t_1\alpha_{i_1} + \cdots + t_l\alpha_{i_l}$ is a multiple of the null root δ, which implies $\{i_1, \dots, i_l\} = I$. Hence $\varepsilon_i(b') = 0$ for all $i \in I$, which contradicts $b' \neq u_\lambda$. Q.E.D.

Note that the subcrystal $D \otimes u_\lambda$ of $D \otimes B(\lambda)$ is isomorphic to the crystal $D \otimes T_\lambda$, where T_λ denotes the crystal with a single element t_λ of weight λ and with $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$. Let B be a regular crystal. In the next theorem, we will show that any morphism of crystals $\Psi : D \otimes u_\lambda \to B$ commuting with the $\tilde{e}_i$'s ($i \in I$) can be extended uniquely to a morphism of regular crystals from $D \otimes B(\lambda) \to B$.

Theorem 4.5 *Let D be a finite regular crystal, B a regular crystal, and λ a dominant integral weight. Suppose that there is a morphism of crystals*

$$\Psi : D \otimes u_\lambda \to B$$

such that $\Psi(D \otimes u_\lambda) \subset B$ and Ψ commutes with the $\tilde{e}_i$'s ($i \in I$).

Then, if rank $\mathfrak{g} > 2$, the map Ψ can be uniquely extended to a morphism of regular crystals

$$\tilde{\Psi} : D \otimes B(\lambda) \to B.$$

Proof. Let Σ be the set of pairs $(S, \tilde{\Psi})$ satisfying the following properties:

$$D \otimes u_\lambda \subset S \subset D \otimes B(\lambda), \tag{4.4}$$

$$\tilde{e}_i^{\max} S \subset S \text{ for any } i \in I, \tag{4.5}$$

$$\tilde{\Psi} \text{ is a map from } S \text{ to } B \text{ such that } \tilde{\Psi}|_{D\otimes u_\lambda} = \Psi, \tag{4.6}$$

$$\mathrm{wt}(\tilde{\Psi}(b)) = \mathrm{wt}(b) \text{ and } \varepsilon_i(\tilde{\Psi}(b)) = \varepsilon_i(b) \text{ for any } b \in S \text{ and } i \in I, \tag{4.7}$$

$$\tilde{\Psi}(\tilde{e}_i^{\max} b) = \tilde{e}_i^{\max} \tilde{\Psi}(b) \quad \text{for any } b \in S \text{ and } i \in I. \tag{4.8}$$

Since Σ is inductively ordered, by Zorn's Lemma, it has a maximal element. Let $(S, \tilde{\Psi})$ be a maximal element. It is enough to prove that S is the same as $D \otimes B(\lambda)$. Assume that they are different.

First we shall prove that there exists $b \in D \otimes B(\lambda) \setminus S$ such that $\tilde{e}_i^{\max}(b) \in S \cup \{b\}$ for any $i \in I$. If it were not true, for any $b \in D \otimes B(\lambda) \setminus S$, there would exist i such that $\tilde{e}_i^{\max}(b) \notin S \cup \{b\}$. Let us take $b_0 \in D \otimes B(\lambda) \setminus S$. Then there is i_0 such that $b_1 = \tilde{e}_{i_0}^{\max}(b) \notin S \cup \{b_0\}$. Repeating this we can find a sequence $\{b_k\}$ and $\{i_k\}$ such that $b_{k+1} = \tilde{e}_{i_k}^{\max}(b_k) \notin S \cup \{b_k\}$. This contradicts Proposition 4.4. Hence there exists $b \notin S$ and $\tilde{e}_i^{\max} b \in S \cup \{b\}$ for all $i \in I$. We shall choose such a b.

Next we shall show that there exists i_0 such that $\tilde{e}_{i_0}^{\max}(b) \in S$. Assuming the contrary, we shall deduce a contradiction. Write $b = b_1 \otimes b_2$. If $\tilde{e}_i^{\max} b = b$ for all $i \in I$, then $0 = \varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle)$. This implies $\varepsilon_i(b_1) = 0$ for every i, and therefore $\langle h_i, \mathrm{wt}(b_1)\rangle = \varphi_i(b_1) \geq 0$. Since $\langle c, \mathrm{wt}(b_1)\rangle = 0$, we have $\langle h_i, \mathrm{wt}(b_1)\rangle = 0$ for every i. Thus we obtain $\varepsilon_i(b_2) = 0$ for every i and hence $b_2 = u_\lambda$. This contradicts $D \otimes u_\lambda \subset S$.

Note that $\varphi_{i_0}(\tilde{\Psi}(\tilde{e}_{i_0}^{\max} b)) = \varphi_{i_0}(\tilde{e}_{i_0}^{\max} b) \geq \varepsilon_{i_0}(b)$. We define $\tilde{\Psi}(b)$ to be $\tilde{f}_{i_0}^{\varepsilon_{i_0}(b)} \tilde{\Psi}(\tilde{e}_{i_0}^{\max} b) \in B$. We will show that $(S \cup \{b\}, \tilde{\Psi})$ satisfies (4.4–8). The properties (4.4–6) are automatically satisfied. For (4.7), note that

$$\mathrm{wt}(\tilde{\Psi}(b)) = \mathrm{wt}(\tilde{\Psi}(\tilde{e}_{i_0}^{\max} b)) - \varepsilon_{i_0}(b)\alpha_{i_0} = \mathrm{wt}(\tilde{e}_{i_0}^{\max} b) - \varepsilon_{i_0}(b)\alpha_{i_0} = \mathrm{wt}(b).$$

We shall show $\varepsilon_i(\tilde{\Psi}(b)) = \varepsilon_i(b)$ for $i \in I$. Set $J = \{i, i_0\} \subsetneq I$. Let K be the connected component of $D \otimes B(\lambda)$ as a $U_q'(\mathfrak{g}_J)$-crystal containing b. Then K is a finite set. Take

a highest weight vector $b_1 \in K \subset D \otimes B(\lambda)$. Then since $\tilde{e}_{i_0}^{\max}(b) \in S$ and $\tilde{e}_i^{\max} S \subset S$ for all $i \in I$, b_1 lies in S. By (4.7), $\tilde{\Psi}(b_1)$ is also a highest weight vector with respect to the J-colored arrows, and $\mathrm{wt}(\tilde{\Psi}(b_1)) = \mathrm{wt}(b_1)$. Hence the map $b_1 \mapsto \tilde{\Psi}(b_1)$ extends to a morphism of $U_q'(\mathfrak{g}_J)$-crystals $\psi : K \to B$. Evidently, $\psi|_{K \cap S} = \tilde{\Psi}|_{K \cap S}$. Since

$$\tilde{\Psi}(b) = \tilde{f}_{i_0}^{\varepsilon_{i_0}(b)} \tilde{\Psi}(\tilde{e}_{i_0}^{\max} b) = \tilde{f}_{i_0}^{\varepsilon_{i_0}(b)} \psi(\tilde{e}_{i_0}^{\max} b) = \psi(\tilde{f}_{i_0}^{\varepsilon_{i_0}(b)} \tilde{e}_{i_0}^{\max} b) = \psi(b),$$

we have the desired property $\varepsilon_i(\tilde{\Psi}(b)) = \varepsilon_i(b)$.

Finally, let us prove (4.8). If $\tilde{e}_i^{\max}(b) \in S$, then

$$\tilde{e}_i^{\max} \tilde{\Psi}(b) = \tilde{e}_i^{\max} \psi(b) = \psi(\tilde{e}_i^{\max}(b)) = \tilde{\Psi}(\tilde{e}_i^{\max} b).$$

If $\tilde{e}_i^{\max}(b) = b$, then $\varepsilon_i(b) = 0$, and hence $\varepsilon_i(\tilde{\Psi}(b)) = 0$. Thus $\tilde{\Psi}(\tilde{e}_i^{\max} b) = \tilde{\Psi}(b) = \tilde{e}_i^{\max} \tilde{\Psi}(b)$.
Q.E.D.

Corollary 4.6 *Let B be a regular crystal with regular head. For an arbitrary connected component $E(b_0)$ of $H(B)$, let $\psi : E(b_0) \hookrightarrow H(B)$ be the inclusion map. Then there exists a unique embedding of regular crystals $\Psi : E(b_0) \otimes B(\lambda_0) \to B$ such that $\Psi(b \otimes u_{\lambda_0}) = \psi(b)$ for any $b \in E(b_0)$.*

Proof. Since $E(b_0)$ is finite, the existence and the uniqueness of Ψ follow immediately from Theorem 4.5. We can also see that Ψ is an embedding by Lemma 4.2.
Q.E.D.

The following theorem describes completely the structure of the regular crystals with regular head.

Theorem 4.7 *Suppose* rank $\mathfrak{g} > 2$. *Then any regular crystal B with regular head has the following decomposition:*

$$B \cong \bigsqcup_D D \otimes B(\lambda_D),$$

where D ranges over the connected components of $H(B)$ and λ_D is a dominant integral weight.

Proof. It suffices to prove that $E(b_0) \otimes B(\lambda)$ is connected for all $b_0 \in H(B)$. This follows from the fact that $H(E(b_0) \otimes B(\lambda)) \cong E(b_0) \otimes u_\lambda$ and $E(b \otimes u_\lambda) = E(b) \otimes u_\lambda \ni b_0 \otimes u_\lambda$ for any $b \in E(b_0)$. Q.E.D.

5 Highest Weight Crystals and Perfect Crystals

Let k, l be positive integers, λ a dominant integral weight of level k, and B_l a *perfect crystal* of level l. The definition and the relevant theory of perfect crystals can be found in [5], [6] and [7]. Consider the tensor product of regular crystals $B(\lambda) \otimes B_l$, where $B(\lambda)$ is the crystal for the integrable highest weight module $V(\lambda)$ over $U'_q(\mathfrak{g})$ with a dominant integral highest weight λ. If $k \geq l$, it is known that $B(\lambda) \otimes B_l$ decomposes into a disjoint union of crystals $B(\mu)$, where μ is a dominant integral weight of level k. In fact, $H(B(\lambda) \otimes B_l)$ is a discrete crystal in this case, and coincides with $u_\lambda \otimes B_l^{\leq \lambda}$, where $B_l^{\leq \lambda} = \{b \in B_l | \; \varepsilon_i(b) \leq \langle h_i, \lambda \rangle \;\text{ for all } i \in I\}$. Hence we have

$$B(\lambda) \otimes B_l \cong \bigoplus_{b \in B_l^{\leq \lambda}} B(\lambda + \mathrm{wt}(b)).$$

See [6] and [7] for details.

In this work, we will concentrate on the case when $k < l$. We first observe:

Proposition 5.1 *The crystal $B(\lambda) \otimes B_l$ has a head and*

$$H(B(\lambda) \otimes B_l) \subset u_\lambda \otimes B_l.$$

Proof. For any $b_1 \otimes b_2 \in B(\lambda) \otimes B_l$, we have $E(b_1 \otimes b_2) \subset E(b_1) \otimes B_l$ and $E(b_1) \otimes B_l$ is a finite set. Hence $B(\lambda) \otimes B_l$ has a head. Now, it is clear that $u_\lambda \otimes B_l$ is stable under $\tilde{e}_i$'s ($i \in I$). Moreover, for any $u \otimes b \in B(\lambda) \otimes B_l$, by applying $\tilde{e}_i$'s repeatedly, we get $\tilde{e}_{i_k} \dots \tilde{e}_{i_1}(u \otimes b) = u_\lambda \otimes b' \in u_\lambda \otimes B_l$ for sufficiently large $k \geq 1$. Hence our assertion follows from Lemma 3.6 (a). Q.E.D.

In the following, we will show that the head $H(B(\lambda) \otimes B_l)$ of $B(\lambda) \otimes B_l$ is isomorphic to the perfect crystal B_{l-k}. Moreover, we will prove that there exists an isomorphism of crystals

$$B(\lambda) \otimes B_l \cong B_{l-k} \otimes B(\lambda'),$$

where λ' is the dominant integral weight of level k determined by the crystal isomorphism

$$B(\lambda) \otimes B_k \cong B(\lambda')$$

given in [6].

In order to give more precise statements, let us recall the theory of coherent families of perfect crystals developed in [5]. Let $\{B_l\}_{l \geq 1}$ be a family of perfect crystals B_l of level l, and set $B_l^{\min} = \{b \in B_l \,|\, \langle c, \varepsilon(b) \rangle = l\}$. Here $\varepsilon(b) = \sum_i \varepsilon_i(b) \Lambda_i$, and we will also use $\varphi(b) = \sum_i \varphi_i(b) \Lambda_i$. By the definition of perfect crystal, ε and φ map $B_l^{\min}$ bijectively to $(P_{\mathrm{cl}}^+)_l \overset{def}{=} \{\lambda \in P_{\mathrm{cl}} \mid \langle h_i, \lambda \rangle \geq 0, \, \langle c, \lambda \rangle = l\}$. We set $J = \{(l, b) \,|\, l \geq 1, \, b \in B_l^{\min}\}$.

Definition 5.2 A crystal B_∞ with an element b_∞ is called a *limit* of $\{B_l\}_{l\ge 1}$ if it satisfies the following conditions:

$$\mathrm{wt}(b_\infty)=0,\ \varepsilon(b_\infty)=\varphi(b_\infty)=0, \tag{5.1}$$

(5.2) for any $(l,b)\in J$, there exists an embedding of crystals

$$f_{(l,b)}: T_{\varepsilon(b)}\otimes B_l\otimes T_{-\varphi(b)}\to B_\infty$$

sending $t_{\varepsilon(b)}\otimes b\otimes t_{-\varphi(b)}$ to b_∞,

$$B_\infty=\bigcup_{(l,b)\in J}\mathrm{Im} f_{(l,b)}. \tag{5.3}$$

If a limit exists, we call $\{B_l\}_{l\ge 1}$ a *coherent* family of perfect crystals. It was proved in [5] that the limit (B_∞, b_∞) is unique up to an isomorphism. Note that we have

$$\langle c,\varepsilon(b)\rangle\ge 0\quad\text{for any } b\in B_\infty.$$

We set $B_\infty^{\min}=\{b\in B_\infty\,|\,\langle c,\varepsilon(b)\rangle=0\}$. Then both ε and φ map $B_\infty^{\min}$ bijectively to $P_{\mathrm{cl}}^0=\{\lambda\in P_{\mathrm{cl}}\,|\,\langle c,\lambda\rangle=0\}$. Moreover, there is a linear automorphism σ of P_{cl}^0 such that $\sigma\varphi(b)=\varepsilon(b)$ for any $b\in B_\infty^{\min}$. We assume further the following condition:

(5.4) σ extends to a linear automorphism σ of P_{cl} such that

$$\sigma\varphi(b)=\varepsilon(b)\quad\text{for any } b\in B_l^{\min}.$$

We conjecture that all the coherent families satisfy this condition. Moreover, σ sends the simple roots to the simple roots, and there exists an element of the Weyl group W such that its induced action on P_{cl}^0 coincides with $\sigma|_{P_{\mathrm{cl}}^0}$.

In the sequel, we fix a coherent family $\{B_l\}_{l\ge 1}$ of perfect crystals satisfying the condition (5.4). For positive integers k and l with $k<l$, let λ be a dominant integral weight of level k and set $\lambda'=\sigma^{-1}\lambda$. Then we have:

Lemma 5.3 *There exists a unique embedding of crystals*

$$\psi: B_{l-k}\to T_\lambda\otimes B_l\otimes T_{-\lambda'}\,.$$

Moreover, we have $\psi(B_{l-k}^{\min})\subset T_\lambda\otimes B_l^{\min}\otimes T_{-\lambda'}$.

Proof. Let us first prove the uniqueness. If $b\in B_{l-k}$ is sent to $t_\lambda\otimes b'\otimes t_{-\lambda'}$, then we have $\varepsilon(b')=\varepsilon(b)+\lambda$, and hence we have $\langle c,\varepsilon(b')\rangle=\langle c,\varepsilon(b)\rangle+k$. Therefore, ψ sends $B_{l-k}^{\min}$ to $T_\lambda\otimes B_l^{\min}\otimes T_{-\lambda'}$, and $\psi|_{B_{l-k}^{\min}}$ is uniquely determined because $\varepsilon: B_l^{\min}\to P_{\mathrm{cl}}$ is injective. Now, the uniqueness of ψ follows from the connectedness of B_{l-k}.

We shall prove the existence. Let us take a dominant integral weight ξ of level $l-k$ and set $\mu=\lambda+\xi$. Then μ is of level l. Set $\mu'=\sigma^{-1}\mu$ and $\xi'=\sigma^{-1}\xi$. Let us take $b_l\in B_l$

such that $\varepsilon(b_l)=\mu$ and $b_{l-k}\in B_{l-k}$ such that $\varepsilon(b_{l-k})=\xi$. Then they are minimal vectors and we have the embeddings

$$\begin{aligned} f_{(l,b_l)} &: T_\mu\otimes B_l\otimes T_{-\mu'}\to B_\infty,\\ f_{(l-k,b_{l-k})} &: T_\xi\otimes B_{l-k}\otimes T_{-\xi'}\to B_\infty \end{aligned}$$

such that $f_{(l,b_l)}(b_l)=f_{(l-k,b_{l-k})}(b_{l-k})=b_\infty$. We shall show

$$\mathrm{Im}(f_{(l-k,b_{l-k})})\subset \mathrm{Im}(f_{(l,b_l)}). \tag{5.5}$$

Since B_{l-k} is connected, it is enough to show that if $b\in B_{l-k}$ satisfies $\tilde e_i(b)\neq 0$ and $f_{(l-k,b_{l-k})}(t_\xi\otimes b\otimes t_{-\xi'})\in \mathrm{Im}(f_{(l,b_l)})$, then $f_{(l-k,b_{l-k})}(t_\xi\otimes \tilde e_i b\otimes t_{-\xi'})$ also belongs to $\mathrm{Im}(f_{(l,b_l)})$. Write $f_{(l-k,b_{l-k})}(t_\xi\otimes b\otimes t_{-\xi'})=f_{(l,b_l)}(t_\mu\otimes b'\otimes t_{-\mu'})$ with $b'\in B_l$. Then we have $\varepsilon_i(t_\xi\otimes b\otimes t_{-\xi'})=\varepsilon_i(t_\mu\otimes b'\otimes t_{-\mu'})$, which implies $\varepsilon_i(b')=\varepsilon_i(b)+\langle h_i,\mu-\xi\rangle>0$. Hence we have $f_{(l-k,b_{l-k})}(t_\xi\otimes \tilde e_i b\otimes t_{-\xi'})=f_{(l,b_l)}(t_\mu\otimes \tilde e_i b'\otimes t_{-\mu'})$, which gives (5.5). Therefore we obtain an embedding of crystal $T_\xi\otimes B_{l-k}\otimes T_{-\xi'}\to T_\mu\otimes B_l\otimes T_{-\mu'}$. This induces the desired embedding ψ. Q.E.D.

Theorem 5.4 *Suppose* rank $\mathfrak{g}>2$, *and let* $\{B_l\}_{l\geq 1}$ *be a coherent family of perfect crystals satisfying the condition* (5.4). *For a pair of positive integers* k *and* l *with* $k<l$, *let* λ *be a dominant integral weight of level* k *and* $\lambda'=\sigma^{-1}\lambda$. *Then we have an isomorphism of crystals*

$$B(\lambda)\otimes B_l\cong B_{l-k}\otimes B(\lambda'). \tag{5.6}$$

Proof. Let $\psi: B_{l-k}\to T_\lambda\otimes B_l\otimes T_{-\lambda'}$ be the embedding given in Lemma 5.3. Let $B_l^{(\lambda)}$ be the subset of B_l such that $\psi(B_{l-k})=T_\lambda\otimes B_l^{(\lambda)}\otimes T_{-\lambda'}$. In order to prove the theorem, we shall show:

(5.7) $H_\lambda=u_\lambda\otimes B_l^{(\lambda)}$ is closed under $\tilde e_i$'s $(i\in I)$,

(5.8) for any $b\in B_l, E(u_\lambda\otimes b)\ni u_\lambda\otimes b'$ for some $b'\in B_l^{(\lambda)}$,

(5.9) there exists a bijection $\Psi: u_\lambda\otimes B_l^{(\lambda)}\to B_{l-k}$ that commutes with $\tilde e_i$'s $(i\in I)$.

Once we have proved them, Lemma 3.6 along with Lemma 3.3 would imply

$$H(B(\lambda)\otimes B_l)=u_\lambda\otimes B_l^{(\lambda)},$$

and, since $H_\lambda\cong B_{l-k}$ is connected, Theorem 4.7 yields a crystal isomorphism

$$B(\lambda)\otimes B_l\cong H_\lambda\otimes B(\lambda')\cong B_{l-k}\otimes B(\lambda').$$

Proof of (5.7) *and* (5.9): They are easily deduced from the existence of ψ and the fact that $\tilde e_i(b)=0$ if and only if $\varepsilon_i(b)=0$ for b in B_{l-k} or in $u_\lambda\otimes B_l^{(\lambda)}$.

Proof of (5.8): Let us take a dominant integral weight ξ of level $l-k$ and set $\mu = \lambda + \xi$. Since B_l is perfect, there exists a unique element $b' \in B_l$ with $\varepsilon(b') = \mu$. Then b' belongs to $B_l^{(\lambda)}$ by Lemma 5.3. We have a crystal isomorphism $B(\mu) \otimes B_l \xrightarrow{\sim} B(\mu')$ given by $u_\mu \otimes b' \mapsto u_{\mu'}$, where $\mu' = \sigma^{-1}\mu$, and u_μ (resp. $u_{\mu'}$) denotes the highest weight vector of $B(\mu)$ (resp. $B(\mu')$) (cf. [6]). Hence, for any $b \in B_l$, there exist $i_1, \ldots, i_t \in I$ such that

$$\tilde{e}_{i_t} \ldots \tilde{e}_{i_1}(u_\mu \otimes b) = u_\mu \otimes \tilde{e}_{i_t} \ldots \tilde{e}_{i_1} b = u_\mu \otimes b'.$$

In particular, we have $\varepsilon_{i_s}(\tilde{e}_{i_{s-1}} \ldots \tilde{e}_{i_1} b) > \langle h_{i_s}, \mu \rangle \geq \langle h_{i_s}, \lambda \rangle$ for $s = 1, \ldots, t$. This gives

$$\tilde{e}_{i_t} \ldots \tilde{e}_{i_1}(u_\lambda \otimes b) = u_\lambda \otimes \tilde{e}_{i_t} \ldots \tilde{e}_{i_1} b = u_\lambda \otimes b' \in u_\lambda \otimes B_l^{(\lambda)},$$

which proves (5.8). Q.E.D.

In the following, we will give a list of coherent families of perfect crystals $\{B_l\}_{l \geq 1}$ satisfying the condition (5.4) for each quantized affine algebra $U_q'(\mathfrak{g})$ of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$. For a positive integer $k < l$, and a dominant integral weight $\lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_n\Lambda_n$ of level k, Theorem 5.4 yields an isomorphism of crystals

$$B(\lambda) \otimes B_l \cong B_{l-k} \otimes B(\lambda'),$$

where $\lambda' = \sigma^{-1}\lambda$. We will also give explicit descriptions of the head $u_\lambda \otimes B_l^{(\lambda)}$ of $B(\lambda) \otimes B_l$, $\lambda' = \sigma^{-1}\lambda$, and the isomorphism $\Psi : u_\lambda \otimes B_l^{(\lambda)} \xrightarrow{\sim} B_{l-k}$. We follow the notations in [5] and [7].

(a) $\mathfrak{g} = A_n^{(1)}$ $(n \geq 2)$:

$$\begin{aligned}
B_l &= \{b = (x_1, \ldots, x_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1} \mid s(b) = \sum_{i=1}^{n+1} x_i = l\}, \\
k &= a_0 + \cdots + a_n, \\
\lambda' &= a_n\Lambda_0 + a_0\Lambda_1 + \cdots + a_{n-1}\Lambda_n, \\
B_l^{(\lambda)} &= \{b = (x_1, \ldots, x_{n+1}) \in B_l \mid x_1 \geq a_0, x_2 \geq a_1, \ldots, x_{n+1} \geq a_n\}.
\end{aligned}$$

As an A_n-crystal, B_l is isomorphic to $B(l\Lambda_1)$. The crystal structure on B_l is described in [5] and [7].

The isomorphism $\Psi : u_\lambda \otimes B_l^{(\lambda)} \xrightarrow{\sim} B_{l-k}$ is given by

$$\Psi(u_\lambda \otimes (x_1, \ldots, x_{n+1})) = (x_1 - a_0, \ldots, x_{n+1} - a_n). \tag{5.10}$$

(b) $\mathfrak{g} = A^{(2)}_{2n-1}$ $(n \geq 3)$:

$$\begin{aligned}
B_l &= \{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \in \mathbb{Z}^{2n}_{\geq 0} |\ s(b) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \bar{x}_i = l\}, \\
k &= a_0 + a_1 + 2(a_2 + \cdots + a_n), \\
\lambda' &= a_1\Lambda_0 + a_0\Lambda_1 + a_2\Lambda_2 + \cdots + a_n\Lambda_n, \\
B_l^{(\lambda)} &= \{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \in B_l | \\
&\qquad x_i, \bar{x}_i \geq a_i\ (i = 2, \dots, n),\ x_1 \geq a_0,\ \bar{x}_1 \geq a_1\}.
\end{aligned}$$

As a C_n-crystal, B_l is isomorphic to $B(l\Lambda_1)$. The crystal structure on B_l is described in [5] and [7].

The isomorphism $\Psi : u_\lambda \otimes B_l^{(\lambda)} \xrightarrow{\sim} B_{l-k}$ is given by

$$\begin{aligned}
&\Psi(u_\lambda \otimes (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1)) \\
&= (x_1 - a_0, x_2 - a_2, \dots, x_n - a_n, \bar{x}_n - a_n, \dots, \bar{x}_2 - a_2, \bar{x}_1 - a_1).
\end{aligned} \tag{5.11}$$

(c) $\mathfrak{g} = B^{(1)}_n$ $(n \geq 3)$:

$$\begin{aligned}
B_l =&\{b = (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \in \mathbb{Z}^{2n+1}_{\geq 0} \ | \\
&\qquad x_0 = 0 \text{ or } 1,\ s(b) = \sum_{i=1}^{n} x_i + x_0 + \sum_{i=1}^{n} \bar{x}_i = l\}, \\
k =&a_0 + a_1 + 2(a_2 + \cdots + a_{n-1}) + a_n, \\
\lambda' =&a_1\Lambda_0 + a_0\Lambda_1 + a_2\Lambda_2 + \cdots + a_n\Lambda_n, \\
B_l^{(\lambda)} =&\{b = (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \in B_l \mid x_1 \geq a_0,\ \bar{x}_1 \geq a_1, \\
&\qquad x_i, \bar{x}_i \geq a_i\ (i = 2, \dots, n-1),\ 2x_n + x_0 \geq a_n,\ 2\bar{x}_n + x_0 \geq a_n\}.
\end{aligned}$$

As a B_n-crystal, B_l is isomorphic to $B(l\Lambda_1)$. The crystal structure on B_l is described in [5] and [7].

The isomorphism $\Psi : u_\lambda \otimes B_l^{(\lambda)} \xrightarrow{\sim} B_{l-k}$ is given as follows. If a_n is even,

(5.12)

$$\begin{aligned}
&\Psi(u_\lambda \otimes (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1)) \\
&= (x_1 - a_0, x_2 - a_2, \dots, x_n - \frac{a_n}{2}, x_0, \bar{x}_n - \frac{a_n}{2}, \dots, \bar{x}_2 - a_2, \bar{x}_1 - a_1).
\end{aligned}$$

If a_n is odd,

$$\begin{aligned}
&\Psi(u_\lambda \otimes (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1)) \\
&= \begin{cases}
(x_1 - a_0, x_2 - a_2, \dots, x_n - \frac{a_n+1}{2}, 1, \bar{x}_n - \frac{a_n+1}{2}, \\
\qquad \bar{x}_{n-1} - a_{n-1}, \dots, \bar{x}_2 - a_2, \bar{x}_1 - a_1) & \text{if } x_0 = 0, \\
(x_1 - a_0, x_2 - a_2, \dots, x_n - \frac{a_n-1}{2}, 0, \bar{x}_n - \frac{a_n-1}{2}, \\
\qquad \bar{x}_{n-1} - a_{n-1}, \dots, \bar{x}_2 - a_2, \bar{x}_1 - a_1) & \text{if } x_0 = 1.
\end{cases}
\end{aligned} \tag{5.13}$$

(d) $\mathfrak{g} = A_{2n}^{(2)}$ $(n \geq 2)$:

$$
\begin{aligned}
B_l =& \{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \in \mathbb{Z}_{\geq 0}^{2n} \mid s(b) = \sum_{i=1}^n x_i + \sum_{i=1}^n \bar{x}_i \leq l\}, \\
k =& a_0 + 2(a_1 + \cdots + a_n), \\
\lambda' =& \lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_n\Lambda_n, \\
B_l^{(\lambda)} =& \{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \in B_l \mid x_i, \bar{x}_i \geq a_i \ (i = 1, \dots, n),\ s(b) \leq l - a_0\}.
\end{aligned}
$$

As a C_n-crystal, B_l is isomorphic to $B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$. The crystal structure on B_l is described in [5] and [7].

The isomorphism $\Psi : u_\lambda \otimes B_l^{(\lambda)} \xrightarrow{\sim} B_{l-k}$ is given by

$$
\begin{aligned}
&\Psi(u_\lambda \otimes (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1)) \\
&= (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n, \bar{x}_n - a_n, \dots, \bar{x}_2 - a_2, \bar{x}_1 - a_1).
\end{aligned}
\tag{5.14}
$$

(e) $\mathfrak{g} = D_{n+1}^{(2)}$ $(n \geq 2)$:

$$
\begin{aligned}
B_l =& \{b = (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \in \mathbb{Z}_{\geq 0}^{2n+1} \mid \\
& \qquad x_0 = 0 \text{ or } 1,\ s(b) = \sum_{i=1}^n x_i + x_0 + \sum_{i=1}^n \bar{x}_i \leq l\}, \\
k =& a_0 + 2(a_1 + \cdots + a_{n-1}) + a_n, \\
\lambda' =& \lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_n\Lambda_n, \\
B_l^{(\lambda)} =& \{b = (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \in B_l \mid\ x_i, \bar{x}_i \geq a_i \ (i = 1, \dots, n-1), \\
& \qquad 2x_n + x_0 \geq a_n,\ 2\bar{x}_n + x_0 \geq a_n,\ s(b) \leq l - a_0\}.
\end{aligned}
$$

As a B_n-crystal, B_l is isomorphic to $B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$. The crystal structure on B_l is described in [5] and [7].

The isomorphism $\Psi : u_\lambda \otimes B_l^{(\lambda)} \xrightarrow{\sim} B_{l-k}$ is given as follows. If a_n is even,

$$
\begin{aligned}
&\Psi(u_\lambda \otimes (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1)) \\
&= (x_1 - a_1, x_2 - a_2, \dots, x_n - \frac{a_n}{2}, x_0, \bar{x}_n - \frac{a_n}{2}, \dots, \bar{x}_2 - a_2, \bar{x}_1 - a_1).
\end{aligned}
\tag{5.15}
$$

If a_n is odd,

$$
\begin{aligned}
&\Psi(u_\lambda \otimes (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1)) \\
&= \begin{cases}
(x_1 - a_1, x_2 - a_2, \dots, x_n - \frac{a_n+1}{2}, 1, \bar{x}_n - \frac{a_n+1}{2}, \\
\qquad \bar{x}_{n-1} - a_{n-1}, \dots, \bar{x}_2 - a_2, \bar{x}_1 - a_1) & \text{if } x_0 = 0, \\
(x_1 - a_1, x_2 - a_2, \dots, x_n - \frac{a_n-1}{2}, 0, \bar{x}_n - \frac{a_n-1}{2}, \\
\qquad \bar{x}_{n-1} - a_{n-1}, \dots, \bar{x}_2 - a_2, \bar{x}_1 - a_1) & \text{if } x_0 = 1.
\end{cases}
\end{aligned}
\tag{5.16}
$$

(f) $\mathfrak{g} = C_n^{(1)}$ $(n \geq 2)$:

$$\begin{aligned}
B_l =& \{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \in \mathbb{Z}_{\geq 0}^{2n} \mid s(b) = \sum_{i=1}^n x_i + \sum_{i=1}^n \bar{x}_i \leq 2l,\ s(b) \in 2\mathbb{Z}\},\\
k =& a_0 + \cdots + a_n,\\
\lambda' =& \lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_n\Lambda_n,\\
B_l^{(\lambda)} =& \{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \in B_l \mid x_i, \bar{x}_i \geq a_i\ (i = 1, \dots, n),\ s(b) \leq 2(l - a_0)\}.
\end{aligned}$$

As a C_n-crystal, B_l is isomorphic to $B(0) \oplus B(2\Lambda_1) \oplus \cdots \oplus B(2l\Lambda_1)$. The crystal structure on B_l is described in [5].

The isomorphism $\Psi : u_\lambda \otimes B_l^{(\lambda)} \xrightarrow{\sim} B_{l-k}$ is given by

(5.17)
$$\begin{aligned}
&\Psi(u_\lambda \otimes (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1))\\
&\quad = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n, \bar{x}_n - a_n, \dots, \bar{x}_2 - a_2, \bar{x}_1 - a_1).
\end{aligned}$$

(g) $\mathfrak{g} = D_n^{(1)}$ $(n \geq 4)$:

$$\begin{aligned}
B_l =& \{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \in \mathbb{Z}_{\geq 0}^{2n} \mid x_n = 0 \text{ or } \bar{x}_n = 0,\ s(b) = \sum_{i=1}^n x_i + \sum_{i=1}^n \bar{x}_i = l\},\\
k =& a_0 + a_1 + 2(a_2 + \cdots + a_{n-2}) + a_{n-1} + a_n,\\
\lambda' =& a_1\Lambda_0 + a_0\Lambda_1 + a_2\Lambda_2 + \cdots + a_{n-2}\Lambda_{n-2} + a_n\Lambda_{n-1} + a_{n-1}\Lambda_n,
\end{aligned}$$

$$B_l^{(\lambda)} = \begin{cases}
\{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \in B_l \mid x_1 \geq a_0,\ \bar{x}_1 \geq a_1, & \\
\qquad x_i, \bar{x}_i \geq a_i\ (i = 2, \dots, n-2),\ x_{n-1}, \bar{x}_{n-1} \geq a_n, & \\
\qquad x_{n-1} + x_n \geq a_{n-1}, \bar{x}_{n-1} + x_n \geq a_{n-1}\} & \text{if } a_{n-1} \geq a_n,\\
\{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \in B_l \mid x_1 \geq a_0,\ \bar{x}_1 \geq a_1, & \\
\qquad x_i, \bar{x}_i \geq a_i\ (i = 2, \dots, n-2),\ x_{n-1}, \bar{x}_{n-1} \geq a_{n-1}, & \\
\qquad x_{n-1} + \bar{x}_n \geq a_n,\ \bar{x}_{n-1} + \bar{x}_n \geq a_n\} & \text{if } a_{n-1} \leq a_n.
\end{cases}$$

As a D_n-crystal, B_l is isomorphic to $B(l\Lambda_1)$. The crystal structure on B_l is described in [5] and [7].

The isomorphism $\Psi : u_\lambda \otimes B_l^{(\lambda)} \xrightarrow{\sim} B_{l-k}$ is given as follows. If $a_{n-1} \geq a_n$,

(5.18)
$$\begin{aligned}
&\Psi(u_\lambda \otimes (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1))\\
&\quad = (x_1 - a_0, x_2 - a_2, \dots, x_{n-2} - a_{n-2},\\
&\qquad x_{n-1} - a_n - (a_{n-1} - a_n - x_n)_+,\ (x_n - a_{n-1} + a_n)_+,\\
&\qquad \bar{x}_n + (a_{n-1} - a_n - x_n)_+,\ \bar{x}_{n-1} - a_n - (a_{n-1} - a_n - x_n)_+,\\
&\qquad \bar{x}_{n-2} - a_{n-2}, \dots, \bar{x}_2 - a_2, \bar{x}_1 - a_1),
\end{aligned}$$

and if $a_{n-1} \le a_n$,

$$
\begin{aligned}
&\Psi(u_\lambda \otimes (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1)) \\
&\quad = (x_1 - a_0, x_2 - a_2, \ldots, x_{n-2} - a_{n-2}, \\
&\qquad x_{n-1} - a_{n-1} - (a_n - a_{n-1} - \bar{x}_n)_+,\ x_n + (a_n - a_{n-1} - \bar{x}_n)_+, \\
&\qquad (\bar{x}_n - a_n + a_{n-1})_+,\ \bar{x}_{n-1} - a_{n-1} - (a_n - a_{n-1} - \bar{x}_n)_+, \\
&\qquad \bar{x}_{n-2} - a_{n-2}, \ldots, \bar{x}_2 - a_2, \bar{x}_1 - a_1).
\end{aligned} \tag{5.19}
$$

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